Unified First Law and some general prescription : A redefinition of surface gravity

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The paper deals with an extensive study of the Unified First Law (UFL) in the Freidmann-Robertson-Walker space-time model. By projecting the UFL along the Kodama vector one can always obtain the second Friedmann equation. Also studying the UFL on the event horizon it is found that clausius relation cannot be obtained from the Unified First Law by projecting it along the tangent to the event horizon as it can be for the trapping horizon. However, it is shown in the present work that clausius relation can be obtained by projecting the unified first law along the Kodama vector on the horizon. The result is found to be true for any horizon. Finally motivated by the Unruh temperature for the Rindler observer, we have redefined the surface gravity and clausius relation is obtained from UFL by projecting it along an analogous Kodama vector.

Keywords : Clausius relation, Hawking temperature, Bekenstein entropy, Kodama vector, Trapping horizon, Unified first law.

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In 1970's Hawking [1] showed that BH is not totally black, rather emits thermal radiations by a combine application of quantum mechanics and general relativity at semi classical level. Interestingly, the temperature of the radiation (known as Hawking temperature) and the entropy of the horizon (known as Bekenstein entropy) have a certain universality in the sense that surface gravity (proportional to Hawking temperature) and horizon area (proportional to Bekenstein entropy) [1, 2] are purely geometric entity characterized by the space-time geometry. Also this entropy and temperature are related to the BH mass through the first law of BH thermodynamics: $dM = TdS$ [3]. Moreover this fantastic discovery gave rise to (i) a speculation for a deep interrelationship between gravity theories and thermodynamics and (ii) a clue to the nature of quantum gravity.

However, the first possibility came true in 1995 when Jacobson [4] derived Einstein equations from clausius relations: $\delta Q = TdS$ for all the local Rindler causal horizon through space-time point ($\delta Q \rightarrow$ the energy flux, $T \rightarrow$ Unruh temperature seen by the accelerated observer just inside the horizon). Subsequently, Padmanabhan [5, 6] was able to show the first law of thermodynamics on the horizon, starting from Einstein equations, for a general static spherically symmetric space-time.

Assuming the Universe as a thermodynamical system, this nice interrelation between Einstein equations and thermodynamic laws has been extended in the context of cosmology. For homogeneous and isotropic FRW model it was found [7] that the Friedmann equations are equivalent to the first law of thermodynamics on the apparent horizon having Hawking temperature $T_A = \frac{1}{2\pi R_A}$ and Bekenstein entropy $S_A = \frac{\pi R_A^2}{G}$ ($R_A =$ geometric radius of the apparent horizon). Then in higher dimensional space-time, this equivalence was established for gravity with the Gauss-Bonnet term and for the Lovelock gravity [7–10].

On the contrary, the situation is totally different for universe bounded by the event horizon (which exists only in accelerating phase of the expansion). Wang *et al* [11] showed that universe bounded by apparent horizon is a perfect thermodynamical system as both 1st and 2nd law of thermodynamics hold for perfect fluid with constant equation of state and holographic dark energy models. However, according to them both the thermodynamical laws failed to satisfy on the event horizon. Then assuming first law, Mazumdar *et al* [12–14] were able to satisfy second law of thermodynamics on the event horizon with some realistic restrictions. In analogy with apparent horizon, the entropy and temperature at the event horizon were chosen as $S_E = \frac{\pi R_E^2}{G}$ and $T_E = \frac{1}{2\pi R_E}$. Later, it was found [15, 16] that the temperature taken on the event horizon (*i.e.* $T_E = \frac{1}{2\pi R_E}$) is not correct and taking the corrected form (*i.e.* $T_E^{(m)} = \frac{R_E}{2\pi R_A^2}$) the

thermodynamics on the event horizon has been studied. It has been shown [17] that for the following two choices

$$\begin{array}{ll} (a) & S_E^{(B)} = \frac{\pi R_E^2}{G}, \quad T_E = T_E^{(g)} = \alpha T_E^{(m)} = \frac{\alpha R_E}{2\pi R_A^2}, \quad \alpha = \frac{\dot{R}_A/R_A}{\dot{R}_E/R_E} \\ (b) & S_E^{(m)} = \beta S_E^{(B)}, \quad T_E = T_E^{(m)}, \quad \beta = \frac{2}{R_E^2}\int R_E^2 \frac{dR_A}{R_A} \end{array} \tag{1}$$

both the thermodynamical laws are satisfied on the event horizon. Also for infinitesimal thermal fluctuation, there is a logarithmic correction to the Bekenstein entropy in the 2nd choice [17].

On the other hand, in the context of dynamical BH, Hayward [18–21] introduced the notion of trapping horizon and proposed a method to deal with thermodynamics associated with a trapping horizon. According to him, for spherically symmetric space-times, Einstein equations can be rewritten in a form termed as "Unified first law". Then projecting this Unified first law (UFL) along a trapping horizon, the first law of thermodynamics was derived. Further, from the point of view of universal thermodynamics we consider our universe as a non-stationary gravitational system and FRW model may be considered as dynamical spherically symmetric space-time. Moreover, in FRW model we have only inner trapping horizon which coincides with the apparent horizon [18–23] and Friedmann equations are equivalent to the UFL on the apparent horizon [22, 26]. Also projection of UFL along the tangent to the apparent horizon gives the clausius relation [22].

Further, there is no preferred time coordinate in an evolving time dependent space-time as there is no longer any (asymptotically time-like) Killing vector field. To resolve this problem, Kodama [27] came forward with a geometrically natural divergence free vector field which exists in any time-dependent spherically symmetric space-time. This vector in the literature is popularly known as Kodama vector, and it identifies a natural time like direction outside a dynamic BH. Also there is a conserved current associated with this vector field [27, 28].

In the present work, we shall study the Unified first law (UFL) on the event horizon for FRW model of the universe. The line element for FRW space-time can be written as [24]

$$\begin{aligned} ds^2 &= h_{ab}dx^a dx^b + R^2 d\Omega_2^2 \\ &= -dt^2 + \frac{a^2}{1-kr^2}dr^2 + R^2 d\Omega_2^2 \end{aligned} \tag{2}$$

where $R = ar$ is the area radius, $h_{ab} = \mathrm{diag}(-1, \frac{a^2}{1-kr^2})$ is the metric on the two-space orthogonal to the spherical symmetry. Using null coordinates (l, m) the above metric can be written as

$$ds^2 = -2\, dl\, dm + R^2 d\Omega_2^2 \tag{3}$$

with

$$\frac{\partial}{\partial l, m} = -\sqrt{2}\left(\frac{\partial}{\partial t} \mp \frac{\sqrt{1-kr^2}}{a}\frac{\partial}{\partial r}\right)$$

as future pointing null vectors.

Now according to Hayward [18–21], the trapping horizon (R_T) is defined as $(\partial_t R)_{R=R_T} = 0$ i.e.

$$R_T = \frac{1}{\sqrt{H^2 + \frac{k}{a^2}}} = R_A \quad , \quad k = 0, \pm 1 \tag{4}$$

For any horizon (having area radius R) the surface gravity is defined as [25]

$$\kappa = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}\, h^{ab}\partial_b R) \tag{5}$$

or in explicit form

$$\kappa = -\left(\frac{R}{R_A}\right)^2 \left(\frac{1 - \dot{R}_A/2HR_A}{R}\right). \tag{6}$$

Now the total energy inside the horizon is a purely geometric quantity, related to the structure of the space-time and to the Einstein's equations [22]. According to Misner and sharp [18–23], the total energy is given by

$$E = \frac{R}{2G}(1 - h^{ab}\,\partial_a R\,\partial_b R) \tag{7}$$

which on simplification gives

$$E = \frac{R^3}{2G}\left(H^2 + \frac{k}{a^2}\right) = \frac{R^3}{2GR_A^2} \tag{8}$$

According to Hayward [18–21], the Unified first law

$$dE = A\psi + WdV \tag{9}$$

is nothing but the rearrangement of the Einstein equations. In the above, A and V stand for the area and volume bounded by the horizon, the work density

$$W = -\frac{1}{2}T^{ab}h_{ab} \tag{10}$$

is regarded as the work done by a change of the horizon and the energy-supply term

$$\Psi_a = T_a^b\partial_b R + W\partial_a R \tag{11}$$

determines the total energy flow ($i.e.$ $\delta Q = A\psi$) through the horizon.

We now introduce the Kodama vector for the present FRW model. It is defined as [27, 29]

$$K^a = \epsilon^{ab}\nabla_b R \tag{12}$$

where ϵ^{ab} is the usual Levi-civita tensor in the 2D radial-temporal plane ($i.e.$ normal to the spherical symmetry). For the present homogeneous and isotropic FRW model

$$\epsilon_{lm} = a(t)(dt)_l \wedge (dr)_m \tag{13}$$

and

$$K^b = \left[-a\left(\frac{\partial}{\partial t}\right)^b + HR\left(\frac{\partial}{\partial r}\right)^b\right] \tag{14}$$

Note that Kodama vector is very similar to the Killing vector $\left(\frac{\partial}{\partial t}\right)^a$ in the de-Sitter space. Also Kodama vector takes the role of the time-like Killing vector (in stationary BH space-time) for dynamical BH and FRW space-time. Further, it can be used as a preferred time evolution vector field in spherically symmetric dynamical systems.

From equations (10) and (11) the explicit form of the work density and energy-supply one form for the present model are

$$W = \frac{1}{2}(\rho - p) \tag{15}$$

and

$$\psi = \left(\frac{\rho + p}{2}\right)\{-HRdt + adr\} \tag{16}$$

Hence we have

$$WdV = 2\pi R^2(\rho - p)\{HRdt + adr\} \tag{17}$$

$$\text{and} \qquad A\psi = 2\pi R^2(\rho + p)\{-HRdt + adr\} \tag{18}$$

Also, from eq. (8) we obtain

$$dE = \frac{1}{2GR_A^3}\left[R^3\left(3HR_A - 2R\dot{R}_A\right)dt + 3R^2R_A a\, dr\right] \tag{19}$$

We shall now show that by projecting the UFL along the Kodama vector gives the second Friedmann equation, in general.

For the above one forms using the scalar product with the Kodama vector we have

$$\left\langle dE, K^b \right\rangle = \frac{-aR^3H}{G}\left(\dot{H} - \frac{k}{a^2}\right) \tag{20}$$

Now,

$$\left\langle A\psi, K^b \right\rangle = 4\pi R^3 H a(\rho + p) \tag{21}$$

$$\text{and} \qquad \left\langle W dV, K^b \right\rangle = 0.$$

Thus, projecting UFL along the Kodama vector gives,

$$\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p) \tag{22}$$

which is nothing but the second Friedmann equation on any arbitrary horizon.

We shall now show that the first Friedmann equation can also be obtained from the Unified first law by projecting it along a vector orthogonal to the Kodama vector namely

$$U^\mu = \left(\frac{R}{a}, \frac{1}{H}, 0, 0\right).$$

Clearly the vector U^μ lies on the radial-temporal plane and it has the following properties:

(i) The vector may be space-like, time-like or null depending on R.

(ii) It is divergence-free in nature ($i.e.$ $\nabla_\mu U^\mu = 0$) and there is a current associated with the vector U^μ given by the relation $\xi^\mu = G^{\mu\nu} U_\nu$. Clearly, the vector ξ^μ is conserved $i.e.$

$$\nabla_\mu \xi^\mu = 0.$$

The scalar product of the individual one-form terms on both sides of the UFL with U^μ gives

$$\langle dE, U^\mu \rangle = \frac{R^2}{2G}\left[\frac{2HR^2}{a}\left(\dot{H} - \frac{k}{a^2}\right) + \frac{3}{aH}\left(H^2 + \frac{k}{a^2}\right)\left\{H^2R^2 + a^2\right\}\right]. \tag{23}$$

$$\langle A\psi, U^\mu \rangle = -\frac{2\pi R^2}{aH}(\rho + p)\left\{H^2R^2 - a^2\right\} \tag{24}$$

and

$$\langle W dV, U^\mu \rangle = \frac{2\pi R^2}{aH}(\rho - p)\left\{H^2R^2 + a^2\right\}. \tag{25}$$

Hence projecting the UFL along U^μ and after some algebra we obtain the first Friedmann equation *i.e.*

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \tag{26}$$

Further, it has been shown in the literature that the first law of BH thermodynamics can be obtained by projecting the UFL along the trapping horizon [18–22] i.e.

$$\langle A\psi, z\rangle = \frac{\kappa}{8\pi G}\langle dA, z\rangle \tag{27}$$

where z is a vector tangential to the trapping horizon.

We shall now show that the situation is not so easy in case of event horizon (EH). The area radius of the EH is given by

$$R_E = a\int_t^\infty \frac{dt}{a} \tag{28}$$

(Note that the improper integral converges for accelerating phase of the Universe).

The normal vector to the null hypersurface $R - a\int_t^\infty \frac{dt}{a} = 0$ is given by $n_a = (-1, a, 0, 0)$, a null vector.

From the property of the null vector, n_a is also tangential to the (null) event horizon hypersurface. Then one can easily see that the clausius relation *i.e.* eq. (27) is not satisfied for the event horizon. Thus the claim [18–22] of obtaining the first law of thermodynamics by projecting the UFL along the tangent is only true for trapping horizon, not for any other horizon.

Note that, the relation (21) gives the rate of energy across the horizon. Thus the energy flux across the event horizon during infinitesimal time dt is

$$dQ = 4\pi H R_E^3(\rho + p)dt \tag{29}$$

or using the second Friedmann equation ($\dot{H} - \frac{k}{a^2} = -4\pi G(\rho + p)$)

$$dQ = -\frac{HR_E^3}{G}\left(\dot{H} - \frac{k}{a^2}\right)dt = \frac{R_E^3}{GR_A^3}\dot{R}_A dt \tag{30}$$

where $R_A = \frac{1}{\sqrt{H^2+\frac{k}{a^2}}}$ has been used in the last equality.
Recently, a notion of generalized Hawking temperature [17] (see eq. (1))

$$T_E^{(G)} = \alpha\frac{R_E}{2\pi R_A^2} \tag{31}$$

has been introduced on the event horizon for the validity of the thermodynamical laws. So in the present context using the first choice in eq. (1) we have

$$T_E^{(G)} dS_E = \frac{R_E^3}{GR_A^3}\dot{R}_A dt \tag{32}$$

Thus we obtain the clausius relation $\delta Q = T_E^{(G)} dS_E$ on the event horizon, by projecting the UFL along the Kodama vector on the horizon. It is interesting to note that the present approach to obtain the clausius relation (*i.e.* the first law of thermodynamics) from the Unified first law is a general prescription and it holds in any horizon even in the trapping horizon. So we have the following conclusions :

(a) Projecting the UFL along the Kodama vector, the second Friedmann equation is always obtained.

(b) Projecting the UFL along a vector orthogonal to Kodama vector (having other properties same as Kodama vector), the first Friedmann equation can be obtained.

(c) Projecting the Unified first law along the tangent to the horizon to obtain the first law of thermodynamics is valid only for the trapping horizon.

(d) First law of thermodynamics on any horizon can be obtained from the Unified first law by projecting it along the Kodama vector on the horizon.

Finally, we redefine the surface gravity motivated by Rindler observer. We have seen that at the local Rindler causal horizon the Unruh temperature is proportional to the acceleration of the free falling observer. So in analogy with the Unruh temperature we assume that the surface gravity should be proportional to the acceleration of the model *i.e.*

$$\kappa = k_0 R \frac{\ddot{a}}{a} = k_0 R \left[\dot{H} + H^2 \right] \tag{33}$$

where k_0 is a dimensionless constant of proportionality and R is introduced on dimensional ground. Then using Einstein equations, the redefined Hawking temperature becomes

$$T_{RH} = -\frac{2k_0 RG}{3}(\rho + 3p) \tag{34}$$

By introducing vector

$$V^a = \left[-\frac{3a}{R} \left(\frac{\partial}{\partial t} \right)^a + H \left(\frac{\partial}{\partial r} \right)^a \right], \tag{35}$$

we obtain

$$\langle dE, V^a \rangle = 4\pi H R^3 (\rho + 3p) a$$

and thus we have the clausius relation

$$\langle dE, V^a \rangle = \langle T_{RH} dS, V^a \rangle$$

$$i.e. \quad \delta Q = T_{RH} dS$$

provided $k_0 = \frac{3}{2}$.
(It is worth mentioning that the vector V^a here also have the properties similar to Kodama vector and is termed as modified Kodama vector.)

Therefore, in the first part we have shown that the Hayward-Kodama definition of surface gravity (defined in eq. (5)) for dynamical model is valid only for apparent horizon in the present context. However, projecting with the Kodama vector we can not only obtain both the Friedmann equations from the UFL but also are able to obtain the clausius relation on the event horizon (or any horizon) with temperature as generalized Hawking temperature. Also the first Friedmann equation can be obtained from UFL by projecting along the orthogal direction of the Kodama vector. Finally, in analogy with Unruh temperature the surface gravity is redefined as proportional to acceleration in FRW model and it is possible to obtain the clausius relation by projecting the UFL along V^a ($i.e.$ modified Kodama vector).

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